

A Structure Theorem for Small Sumsets in Nonabelian Groups

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February 28, 2012

Abstract

Let G be an arbitrary finite group and let S and T be two subsets such that $|S| \geq 2$, $|T| \geq 2$, and $|TS| \leq |T| + |S| - 1 \leq |G| - 2$. We show that if $|S| \leq |G| - 4|G|^{1/2}$ then either S is a geometric progression or there exists a non-trivial subgroup H such that either $|HS| \leq |S| + |H| - 1$ or $|SH| \leq |S| + |H| - 1$. This extends to the nonabelian case classical results for abelian groups. When we remove the hypothesis $|S| \leq |G| - 4|G|^{1/2}$ we show the existence of counterexamples to the above characterization whose structure is described precisely.

1 Introduction

Let $(G, +)$ be a finite abelian group written additively. Let S be a subset of G such that $T + S \neq G$ and

$$|T + S| \leq |T| + |S| - 2 \tag{1}$$

for some subset T of G . A Theorem of Mann [15] says that S must be well covered by cosets of a subgroup. More precisely, there must exist a proper subgroup H of G such that

$$|S + H| \leq |S| + |H| - 2.$$

Mann's Theorem can be thought of as simplified, or one-sided, version of Kneser's Theorem [14] which gives a structural result for the pair of subsets $\{S, T\}$ rather than a single subset. If one weakens the condition (1) to

$$|T + S| \leq |T| + |S| - 1 \tag{2}$$

for some set T such that $|S + T| \leq |G| - 2$, then a structural change occurs because the sets S and T can be arithmetic progressions and not well covered by cosets. However, this is the only alternative i.e. if $|T| \geq 2$ and S is not an arithmetic progression, then a simple, one-sided, version of the Kempermann Structure Theorem [13] says that there must exist a proper subgroup such that

$$|S + H| \leq |S| + |H| - 1. \tag{3}$$

In the present work we are interested in the nonabelian counterpart of the above results. Caution is in order because the two-sided abelian additive theorems do not seem to generalize. In particular counter-examples to the intuitive nonabelian generalization of Kneser's Theorem

were found by Olson [18] and the second author [21]. However, Mann's theorem was generalized to the nonabelian setting [21, 5]. It was obtained that, if S is a subset of a finite group (G, \times) (from now on written multiplicatively to emphasize that G is not necessarily abelian) for which there is a subset T such that $TS \neq G$ and

$$|TS| \leq |T| + |S| - 2,$$

then there must exist a proper subgroup H such that S is well-covered by either left or right cosets modulo a subgroup H , i.e. we have

$$\text{either } |SH| \leq |S| + |H| - 2 \text{ or } |SH| \leq |H| + |S| - 2.$$

Note that the difference with the abelian case is that we cannot control whether S is covered by left or right cosets.

Our main result is to obtain a structural result on S under a generalization of (2) to nonabelian groups. Specifically, we prove:

Theorem 1. *Let S be a subset of a finite group G for which there exists a subset T such that $2 \leq |T|$ and $|TS| \leq \min(|G| - 2, |T| + |S| - 1)$. Then one of the following holds*

- (i) *S is a geometric progression, i.e. there exist $g, a \in G$ such that one of the two sets gS, Sg equals $\{1, a, a^2, \dots, a^{|S|-1}\}$,*
- (ii) *there exists a proper subgroup H of G such that*

$$|HS^\varepsilon| \leq |H| + |S| - 1$$

where S^ε denotes either S or S^{-1} .

- (iii) *there exists a subgroup H and an element a of G such that $|HaH| = |H|^2$ and, letting $A = H \cup Ha$,*

$$|AS^\varepsilon| = |A| + |S| - 1 = |G| - |A|.$$

Note that property (iii) collapses to a particular case of (i) if the group G is abelian, since then we can only have $H = \{1\}$. Condition $|HaH| = |H|^2$ in (iii) also implies that it can only occur for subsets S of G that are quite close to being the whole group, since we must clearly have $|H| \leq |G|^{1/2}$ and $|S| = |G| + 1 - 4|H|$, in other words:

Corollary 2. *Let S be a subset of a finite group G for which there exists a subset T such that $2 \leq |T|$ and $|TS| \leq \min(|G| - 2, |T| + |S| - 1)$ and such that $|S| \leq |G| - 4|G|^{1/2}$: then*

- *either S is a geometric progression,*
- *or there exists a proper subgroup H of G such that*

$$|HS^\varepsilon| \leq |H| + |S| - 1.$$

The condition $|S| \leq |G| - 4|G|^{1/2}$ in Corollary 2 is unlikely to be improved upon asymptotically, for we shall show in the final section that, assuming a number-theoretic conjecture (the existence of an infinite number of Sophie Germain primes), there exist infinite families

of groups G with subsets S such that $|S| \leq |G| - O(\sqrt{|G|})$ and that satisfy the hypothesis of Corollary 2 but not its conclusion.

We shall use Hamidoune's atomic method to derive Theorem 1. If S is a generating subset containing 1 of a finite group, then A is a k -atom of S if it is of minimum cardinality among subsets X such that $|X| \geq k$, $|XS| \leq |G| - k$ and $|XS| - |X|$ is of minimum possible cardinality (see Section 2 for detailed definitions). The isoperimetric or atomic method was introduced by Hamidoune in [4] and the study of k -atoms was used in a number of papers including [10, 12] to give structural results in abelian groups on sets S, T such that $|S + T| \leq |S| + |T| + m$. The study of k -atoms for $k = 1, 2$ has also yielded generalizations to nonabelian groups of several addition theorems [21, 4, 8, 5, 11]. The structure of 2-atoms is again crucial to our present study. Under the hypothesis of Theorem 1, when a 2-atom of S or S^{-1} has cardinality 2, it yields property (i). When a 2-atom of S or S^{-1} is a subgroup, it yields property (ii). It follows from results in [6] that in the abelian case, and in some particular nonabelian cases, under the hypothesis of Theorem 1, any 2-atom containing the unit element must either have cardinality 2 or be a subgroup. However, deriving a similar result in the general nonabelian case has not been attempted until recently, when the topic has attracted the attention of a number of researchers in the area of so-called approximate groups, see e.g. [19].

In one of his last preprints [7], Hamidoune shows that, under the hypothesis of Theorem 1, if (i) and (ii) do not hold then any 2-atom of S containing 1 must be of the form $A = H \cup Ha$ for some subgroup H . He does not try to find out however, whether these particular 2-atoms can actually exist, and our original contribution is to show that, somewhat surprisingly, this last case can indeed occur, but only just, namely only if HS^ε consists of exactly all but two right H -cosets of G .

The paper is organised as follows: in the Section 2 we introduce the isoperimetric tools that we shall need in the sequel. We then proceed in Section 3 to obtain a general upper bound for the cardinality of nonperiodic 2-atoms, thus extending an analogous result obtained by Hamidoune [6] for abelian groups, for normal sets in simple groups by Arad and Muzychuk [1] and in [2] for torsion-free groups. Section 4 contains a somewhat shortened account of Hamidoune's result on 2-atoms obtained in [7]. We then go on to study the particular case of 2-atoms equal to the union of two cosets with the purpose of showing that this last case can mostly not exist, except in an exceptional degenerate case. Our main method will be to show that the exceptional 2-atom defines an edge-transitive graph on cosets of a subgroup of G and a close study of the edge-connectivity of this graph will rule out all subsets S but the ones described above. We conclude by exhibiting a family of examples that show that the leftover case can indeed occur.

2 Notation and preliminary results

Let G be a finite group and let $S \subset G$ be a generating subset containing the unit element 1. For X a subset of G we shall write

$$\partial_S X = XS \setminus X$$

and $X^{*S} = G \setminus (X \cup \partial_S X)$. When the set S is implicit and no confusion can occur we simply write ∂X and X^* .

We shall say that S is *k-separable* if there exists $X \subset G$ such that $|X| \geq k$ and $|X^*| \geq k$. Suppose that S is *k-separable*. The *k-th isoperimetric number* of S is defined by

$$\kappa_k(S) = \min\{\partial X \mid |X| \geq k, |X^*| \geq k\}.$$

For a *k-separable* set S , a subset X achieving the above minimum is called a *k-fragment* of S . A *k-fragment* with minimal cardinality is called a *k-atom*.

Comments. The above definitions can be formulated, as usually done by Hamidoune, in the more general context of directed graphs [6] where sets X are subsets of vertices, ∂X (the boundary of X) is the set of neighbouring vertices of X not in X , and X^* is the complement i.e. the set of vertices neither in X nor ∂X . We are dealing with the case of a Cayley graph G with set of generators equal to the non unit elements of S . The following properties are straightforward and will be used throughout.

- If $A \subseteq B$ then $B^* \subseteq A^*$.
- If S is *k-separable*, then $1 \leq \kappa_{k-1}(S) \leq \kappa_k(S)$. We also have $\kappa_k(S) = \kappa_k(S^{-1})$.
- For any $s \in S^{-1}$, $\kappa_k(S) = \kappa_k(Ss)$ and *k-fragments* (resp. *k-atoms*) of S are *k-fragments* (resp. *k-atoms*) of Ss .
- If F is a *k-fragment* of S , then F^* is a *k-fragment* of S^{-1} .
- If F is a *k-fragment* (resp. *k-atom*) of S then every left translate gF is a *k-fragment* (resp. *k-atom*) of S for any group element g .

Finally we denote by $\alpha_k(S)$ the size of a *k-atom* of S .

The following two theorems obtained by Hamidoune [6] are basic pieces of the atomic method. The first two are intersection properties of atoms.

Theorem 3 (The intersection property for atoms). *Let S be a *k-separable* subset of the group G . Let A and B be two distinct *k-atoms* of S . If $|G| \geq 2\alpha_k(S) + \kappa_k(S)$, then $|A \cap B| \leq k - 1$.*

Note that if $\alpha_2(S) \leq \alpha_2(S^{-1})$ then we have $|G| \geq 2\alpha_k(S) + \kappa_k(S)$ and the intersection property holds for atoms of S . Therefore, if the intersection property does not hold for the atoms of S then it holds for the atoms of S^{-1} . If $\alpha_2(S) \leq \alpha_2(S^{-1})$ the following more general result holds:

Theorem 4 (The fragment-atom intersection property). *Let S be a *k-separable* subset of the group G . Suppose $\alpha_2(S) \leq \alpha_2(S^{-1})$ and let A and F be a *k-atom* and a *k-fragment* of S respectively. Then*

- either $A \subset F$
- or $|A \cap F| \leq k - 1$.

The intersection Theorem 4 together with the fact that left translates of an atom are atoms implies that either 1-atoms containing 1 of S or 1-atoms containing 1 of S^{-1} are subgroups.

Theorem 5 ([3]). *Let S be a 1-separable generating set of a group G with $1 \in S$ and $\alpha_1(S) \leq \alpha_1(S^{-1})$. The atom of S containing 1 is a subgroup*

Note that without loss of generality we can add to the hypothesis of Theorem 1 that S contains 1 (if not replace S by a right translate of S) and that S generates G (if S is not 1-separable in the subgroup that it generates, then part (ii) of Theorem 1 holds trivially). In that case the hypothesis of Theorem 1 translates to: $\kappa_2(S) \leq |S| - 1$.

In the abelian case it was proved in [6] (see also [12]) that 2-atoms that are not subgroups have cardinality at most $\kappa_2(S) - |S| + 3$. This implies in particular that if $\kappa_2(S) \leq |S| - 1$ then 2-atoms are either subgroups or of cardinality 2. In turn this gives that under the hypothesis (2) either S is an arithmetic progression (2-atoms are of cardinality 2) or 2-atoms are subgroups which yields (3).

In the general, non abelian case, it was obtained in [1] in the special case of simple groups and normal sets S that the cardinality of 2-atoms is at most $\kappa_2(S) - |S| + 3$. In the next section we prove in all generality for all finite groups, that if $|G| \geq 2\alpha_2(S) + \kappa_2(S)$, then either $\kappa_2(S) - |S| + 3$ or 2-atoms A of S are *left-periodic*, meaning that there exists group elements $x \neq 1$ such that $xA = A$. Note once more that if the condition $|G| \geq 2\alpha_2(S) + \kappa_2(S)$ does not hold for S , then it necessarily holds for S^{-1} .

3 Nonperiodic 2-atoms

In what follows G is a finite group and S is a 2-separable generating set of G with $1 \in S$ and $|S| \geq 3$. We moreover assume that $|G| \geq 2\alpha_2(S) + \kappa_2(S)$. The purpose of this section is to prove that a 2-atom U of S which is not left-periodic has cardinality at most $|U| \leq \kappa_2(S) - |S| + 3$, Proposition 10 below.

Lemma 6. *Let A be a 2-atom of S and let $H \subset G$ be its maximal left period, i.e. the maximal subgroup such that $HA = A$. Then,*

$$|A \cap Ag| \leq |H| \text{ for all } g \in G \setminus \{1\}. \quad (4)$$

In other words, $A \cap Ag$ contains at most a single coset. In particular, if $g \in H \setminus \{1\}$ then $A \cap Ag \subseteq H$.

Proof. The statement trivially holds if $A = H$. Suppose that

$$A = Ha_1 \cup \cdots \cup Ha_t,$$

is the union of $t \geq 2$ different cosets of H , where $a_1 = 1$. Since $A \neq gA$ for each $g \notin H$, the intersection property of A now says

$$|A \cap gA| \leq 1 \text{ for all } g \in G \setminus H. \quad (5)$$

Consider first that $g \in H \setminus \{1\}$. By the intersection property (5) of A we have $|A \cap a_i A| \leq 1$ for each $i > 1$. According to the decomposition of A into right cosets of H , this means

$$a_i H \cap Ha_i = \{a_i\} \text{ and } a_i H \cap Ha_j = \emptyset, \text{ for } j \neq i. \quad (6)$$

which implies

$$Ha_i g = Ha_j \text{ if and only if } a_i = a_j = a_1,$$

because $a_i g a_j^{-1} \in a_i H a_j^{-1} \cap H$ and $g \neq 1$. It follows that $A \cap Ag = H$.

Suppose now that $g \in G \setminus H$. Let $A = A_1 \cup \dots \cup A_s$ be the decomposition of A into maximal right g -progressions, $A_i = \{w_i, w_i g, \dots, w_i g^{\ell_i-1}\}$. We may assume $|A_1| \geq |A_i|$ for each i and $w_1 = 1$. By the intersection property of A we have

$$|A_1 \cap A_1 g| = |A_1 \cap g A_1| \leq |A \cap g A| \leq 1,$$

which implies $|A_1| \leq 2$. For each A_i such that $w_i \in G \setminus H$ we have

$$|A_1 \cap w_i^{-1} A_i| \leq |A \cap w_i^{-1} A| \leq 1,$$

which implies $|A_i| = 1$. Hence $(A \setminus H)g \cap A = \emptyset$ and thus $A \cap Ag \subseteq Hg$. This completes the proof. \square

Corollary 7. *Let A be a 2-atom of S . If A is not left-periodic then, for each $g \in G \setminus \{1\}$ we have*

$$\max\{|A \cap Ag|, |A \cap gA|\} \leq 1.$$

Proof. That $|A \cap gA| \leq 1$ is the intersection property. That $|A \cap Ag| \leq 1$ is Lemma 6 since if A is not left-periodic then its maximal period is $H = \{1\}$. \square

Lemma 8. *Let A be a k -atom of S , $k \leq 2$. If A is not left-periodic then $|A| \leq \min\{2, |S| - 1\}$.*

Proof. Suppose that $|A| > 2$. Note that, for every element $a \in A$ there is $s \in S \setminus \{1\}$ such that $as^{-1} \in A$, since otherwise $|(A \setminus \{a\})S| - |A \setminus \{a\}| = |AS| - |A|$ contradicting the minimality of the k -atom.

We may therefore define a map $f : A \rightarrow S \setminus \{1\}$ which assigns to each a an $s \neq 1$ such that $as^{-1} \in A$. This map is injective otherwise $|A \cap As^{-1}| \geq 2$, contradicting Corollary 7. \square

Our last preliminary step shows that an aperiodic 2-atom with more than two elements (actually any set satisfying the intersection property) has a large 2-connectivity.

Lemma 9. *Let A be a 2-atom of S with $1 \in A$ and $|A| \geq 3$. If A is not left-periodic then A is 2-separable, and $\kappa_2(A) = 2|A| - 3$. In particular, $\kappa_1(A) = |A| - 1$.*

Proof. Let K be the subgroup generated by A . Since $|A| \geq 3$ there are two distinct elements $a, a' \in K \setminus \{1\}$. Thus, by the intersection property of A ,

$$|K| \geq |\{1, a, a'\}A| \geq |A| + |aA \setminus A| + |a'A \setminus (A \cup aA)| \geq 3|A| - 3.$$

Similarly, we have $|XA| - |X| \geq 2|A| - 3$ for each subset $X \subset K$ with cardinality 2. Moreover there is equality if $1 \in X$ and $X \subset A$. By choosing such an X we have

$$|K| - |XA| \geq (3|A| - 3) - (2|A| - 1) = |A| - 2.$$

Therefore A is 2-separable unless $|A| = 3$ and $|K| = 6$. In this case, however, the six left translates of A form a system of triples on a set of six points, every two of which intersect

in at most one point. Such a structure does not exist. Hence A is 2-separable and $\kappa_2(A) \leq |AX| - |X| = 2|A| - 3$. We next show that $\kappa_2(A) = 2|A| - 3$.

Suppose on the contrary that $\kappa_2(A) \leq 2|A| - 4$. Let B be a 2-atom of A containing 1. Since we have $\kappa_2(A) = \kappa_2(A^{-1})$, we may assume that $\alpha_2(A) \leq \alpha_2(A^{-1})$, otherwise proceed by replacing A by A^{-1} after noticing that A^{-1} must also satisfy Corollary 7. We have $|B| \geq 3$ and

$$3|A| - 3 \leq |BA| \leq 2|A| + |B| - 4, \quad (7)$$

where the leftmost inequality uses the intersection property of A and the rightmost one the fact that B is a 2-atom of A . This implies

$$|A| + 1 \leq |B|, \quad (8)$$

which improves (7) to

$$|A|(|A| + 1)/2 \leq |BA| \leq 2|A| + |B| - 4, \quad (9)$$

and (8) to

$$|A|(|A| - 3)/2 + 4 \leq |B|. \quad (10)$$

Moreover, by Lemma 8 applied to B and A , the 2-atom B is left-periodic. Let $H \subset K$ be the stabilizer of B by left translations. The subgroup H is nontrivial and B is a union of right-cosets of H .

Suppose that $B = H$. Since A generates K there is an $a \in A \setminus H$ and therefore

$$|BA| \geq |B \cup Ba| = 2|B|. \quad (11)$$

By inserting this inequality in the left hand side of (7) we get $|B| \leq 2|A| - 4$, contradicting (10). Therefore we may assume that

$$B = Hb_1 \cup \cdots \cup Hb_t,$$

is the union of $t \geq 2$ different cosets of H , where $b_1 = 1$.

Suppose that $t \geq 3$. Let $1, a, a'$ be three elements in A . By (4), we have

$$|BA| \geq |B \cdot \{1, a, a'\}| \geq 3|B| - 3|H| \geq 2|B|,$$

which, by the same reasoning as in (11), leads to a contradiction. Thus $t = 2$. It follows that $|BA| = 3|H|$ since otherwise we again have $|BA| \geq 2|B|$.

Suppose that $|A| \geq 4$. Let $1, a, a', a''$ be four points in A . The four right translates B, Ba, Ba', Ba'' must each consist of two right cosets of H chosen among the three right H -cosets of BA . But by (4) they must be distinct: this is not possible, so we are left with the case $|A| = 3$.

In this case (7) implies $|H| = |BA| - |B| \leq 2|A| - 4 = 2$ and $|BA| = 6$.

Let us set $H = \{1, h\}$, $B = H \cup Hb$, $BA = H \cup Hb \cup Hc$ and $A = \{1, a, a'\}$. By (4) we have, without loss of generality,

$$Ba = H \cup Hc \quad (12)$$

$$Ba' = Hb \cup Hc \quad (13)$$

The intersection property applied to A implies that $a \notin H$, and $Ha \neq Ha'$, hence (12) implies $Ha = Hc$ and $Hba = H$ i.e. $Hb = Ha^{-1}$, and (13) implies $Ha' = Ha^{-1}$ from which we have $a' = ha^{-1}$ otherwise A is a geometric progression and cannot satisfy the intersection property. Equality (13) also implies $Ha^{-1}a' = Ha$ from which we get $a' = aha$ since we cannot have $a' = a^2$ because A would again be a geometric progression. From this and $a' = ha^{-1}$ we get $a^{-1} = haha$ from which $a^3 = 1$ and $ah = ha$. Therefore B and A together generate a group of order 6 inside which the 6 left translates of A must be distinct and intersect in at most 1 element. This is not possible. \square

We are now ready to prove our upper bound on the size of aperiodic 2-atoms.

Proposition 10. *Let A be a 2-atom of S with $|A| \geq 3$. If A is not left-periodic then $|A| \leq \kappa_2(S) - |S| + 3$.*

Proof. Set $m = \kappa_2(S) - |S|$. We may assume that $1 \in A$. Moreover $\kappa_2(S) \geq |S| - 1$ since otherwise $\kappa_1(S) = \kappa_2(S) < |S| - 1$ and A is a subgroup. By Lemma 9, A is 2-separable and $\kappa_2(A^{-1}) = \kappa_2(A) = 2|A| - 3$.

Suppose first that A generates the same group G as S . Then, S is a witness that $\kappa_2(A^{-1}) \leq |A| + m$, since $|S^{-1}A^{-1}| - |S^{-1}| = |AS| - |S| \leq |A| + m$. By combining the two inequalities for $\kappa_2(A)$ we get $|A| \leq m + 3$ as claimed.

Suppose now that A generates a proper subgroup H of G . Let $S = S_1 \cup \dots \cup S_k$ be the right-decomposition of S modulo H , namely, each S_i is the nonempty intersection of S with a right-coset of H .

If there is an S_i such that $|S_i| \geq 2$ and $|AS_i| \leq |H| - 2$, say $i = 1$, then

$$|A| + |S| + m = |AS| = \sum_i |AS_i| \geq |AS_1| + |S \setminus S_1|,$$

so that $\kappa_2(A) \leq |A| + m$ and the above argument applies.

Suppose that $|S_i| = 1$ or $|AS_i| \geq |H| - 1$ for each i . Let t be the number of i 's such that $|AS_i| \leq |H| - 1$. By Lemma 9, we have $\kappa_1(A) = |A| - 1$, which gives

$$|A| + |S| + m = |AS| = \sum_i |AS_i| \geq |S| + t(|A| - 1), \quad (14)$$

We cannot have $t = 0$ since otherwise $AS = HS$ and $|HS| - |H| < |AS| - |A|$, contradicting that A is a 2-atom. If $t = 1$, then either $|S_1| = 1$ or $|AS_1| = |H| - 1$. In the first case, by taking $X = \{1, x\} \subset H$ we have $|XS| - |X| \leq |AS| - |A|$, contradicting again that A is a 2-atom. In the second case we have $|AS| = |HS| - 1$ and, since $|AS| - |A| \leq |HS| - |H|$, we get $|A| \geq |H| - 1$, which is incompatible with the intersection property of A . Hence $t \geq 2$ and (14) gives $|A| \leq m + 2$. \square

4 Periodic 2-atoms

Throughout the section G is a finite group and S is a 2-separable generating set of G with $1 \in S$ and $|S| \geq 3$.

In this section we show that, if the 2-atom of S is left-periodic and $\kappa_2(S) \leq |S| - 1$, then either there is a subgroup which is a 2-fragment or the 2-atom is the union of at most two cosets of a subgroup. This fact is already stated in the preprint of Hamidoune [7, Theorem 8.1] when $\kappa_2(S) = |S| - 1$, deduced from a more general result in the setting of vertex transitive graphs. We trace back the argument and give a direct proof which is slightly simpler.

If $\kappa_1(S) < |S| - 1$ then $\kappa_2(S) = \kappa_1(S)$ and, by Theorem 5, the atom of S containing 1 is a subgroup. We thus may assume that $\kappa_2(S) = \kappa_1(S) = |S| - 1$.

Following Hamidoune we shall use the following diagram which is useful in the coming arguments. Let $\mathcal{F} = \{F_i, i \in I\}$ be a collection of 2-fragments of S . Each F_i induces the partition $\{F_i, \partial F_i, G \setminus F_i S\}$ of G , where $F_i^* = G \setminus F_i S$ is a 2-fragment of S^{-1} and $\partial^{-1} F_i^* = \partial F_i$. For every pair $F_i, F_j \in \mathcal{F}$ we consider the common refinement of these partitions and use the notation

$$\beta_{ij} = |F_i \cap \partial F_j| \text{ and } \beta'_{ij} = |\partial F_i \cap F_j^*|, \quad i \neq j,$$

and

$$\gamma_{ij} = \gamma_{ji} = |\partial F_i \cap \partial F_j|,$$

which is illustrated in Figure 1.

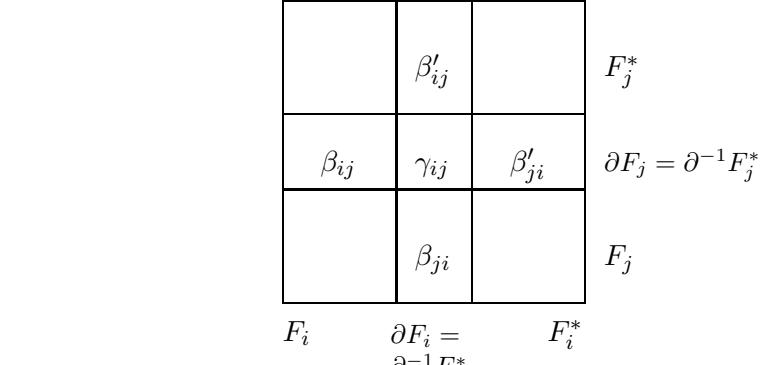


Figure 1: The partition induced by $(F_i, \partial F_i, F_i^*)$ and $(F_j, \partial F_j, F_j^*)$.

We start with a technical lemma which is a simple variant of the intersection property in the case when $\kappa_2(S) = \kappa_1(S)$.

Lemma 11. *Suppose $\kappa_2(S) = \kappa_1(S)$. Let F_1 and F_2 be two 2-fragments of S . Suppose $F_1 \cap F_2 \neq \emptyset$. Then with the notation of Figure 1 we have*

$$\beta_{12} \geq \beta'_{12}.$$

Suppose furthermore that $F_1^* \cap F_2^* \neq \emptyset$. Then $\beta_{12} = \beta'_{12}$, $\beta_{21} = \beta'_{21}$ and

- (i) either $F_1 \cap F_2$ is a 2-fragment of S ,
- (ii) or $|F_1 \cap F_2| = 1$.

Proof. We have

$$\beta_{12} + \gamma_{12} + \beta_{21} \geq |\partial(F_1 \cap F_2)| \geq \kappa_1 = \kappa_2 = \beta'_{12} + \gamma_{12} + \beta_{21}, \quad (15)$$

which implies $\beta_{12} \geq \beta'_{12}$. If both $F_1 \cap F_2$ and $F_1^* \cap F_2^*$ are non empty, then summing (15) together with

$$\beta'_{12} + \gamma_{12} + \beta'_{21} \geq |\partial(F_1^* \cap F_2^*)| \geq \kappa_1 = \beta_{12} + \gamma_{12} + \beta_{21}$$

we obtain that all inequalities must in fact be equalities, meaning that

$$|\partial(F_1 \cap F_2)| = |\partial(F_1^* \cap F_2^*)| = \kappa_1 = \kappa_2$$

which implies the result. \square

We consider two cases according to whether or not $|G| < 2\alpha_2(S^\varepsilon) + \kappa_2(S^\varepsilon)$ holds for $\varepsilon = 1$ or $\varepsilon = -1$.

Lemma 12. *Suppose $\kappa_2(S) = \kappa_1(S)$ and either $|G| < 2\alpha_2(S) + \kappa_2(S)$ or $|G| < 2\alpha_2(S^{-1}) + \kappa_2(S^{-1})$. Then there is a 2-fragment of S or a 2-fragment of S^{-1} which is a subgroup.*

Proof. Let us suppose $|G| < 2\alpha_2(S^{-1}) + \kappa_2(S^{-1})$, the case $|G| < 2\alpha_2(S) + \kappa_2(S)$ being similar. Let F_1, F_2 be any 2-fragments of S such that F_1^* and F_2^* are 2-atoms of S^{-1} . We shall show that $F_1 \cap F_2 = \emptyset$. It will follow that one such 2-fragment containing 1 satisfies $x^{-1}F = F$ for each $x \in F$ and thus F is a subgroup, as claimed.

Suppose on the contrary that $F_1 \cap F_2 \neq \emptyset$. By Lemma 11 we have $\beta_{12} \geq \beta'_{12}$.

Now the hypothesis $|G| < 2\alpha_2(S^{-1}) + \kappa_2(S^{-1})$ translates into $|F_1^*| > |F_1|$ since F_1^* is a 2-atom of S^{-1} (see the remark after Theorem 3).

We have (see Figure 1)

$$|F_1| = |F_1 \cap F_2| + \beta_{12} + |F_1 \cap F_2^*|$$

and

$$|F_1^*| = |F_2^*| = |F_1^* \cap F_2^*| + \beta'_{12} + |F_1 \cap F_2^*|$$

from which $|F_1^*| > |F_1|$ implies, since $\beta_{12} \geq \beta'_{12}$,

$$|F_1^* \cap F_2^*| > |F_1 \cap F_2| > 0.$$

Since F_1^* is a 2-atom of S^{-1} , it follows that

$$\beta'_{12} + \gamma_{12} + \beta'_{21} \geq |\partial(F_1^* \cap F_2^*)| > \kappa_2(S) = \beta_{12} + \gamma_{12} + \beta'_{12},$$

which implies $\beta'_{12} > \beta_{12}$, a contradiction. \square

According to the last lemma we can restrict ourselves to the case when we simultaneously have $|G| \geq 2\alpha_k(S) + \kappa_k(S)$ and $|G| \geq 2\alpha_k(S^{-1}) + \kappa_k(S^{-1})$.

Lemma 13. *Assume that $\kappa_2(S) = \kappa_1(S) = |S| - 1$ and $\alpha_2(S) > 2$ and $\alpha_2(S^{-1}) > 2$. If $|G| \geq 2\alpha_k(S) + \kappa_k(S)$ and $|G| \geq 2\alpha_k(S^{-1}) + \kappa_k(S^{-1})$, then there is a 2-atom of S or of S^{-1} which is the union of at most two right cosets of H .*

Proof. Let A be a 2-atom of S with $1 \in A$. Since $\alpha_2(S) > 2$, by Proposition 10, A is left-periodic. Suppose that A is the union of at least three cosets of a subgroup H . We may assume H to be the stabilizer of A by left translations. Thus there are $x, y \in A$ such that $A_1 = A, A_2 = x^{-1}A, A_3 = y^{-1}A$ are pairwise distinct and contain 1. By the intersection property, $A_1 \cap A_2 \cap A_3 = \{1\}$.

Applying the first part of Lemma 11 with $F_i = A_i$ we obtain that $|A_1 \cap A_2| = 1$ implies

$$|A_1^* \setminus A_2^*| \leq |A_1| - 1. \quad (16)$$

We observe that $A_1^* \setminus A_2^* \neq \emptyset$ since otherwise $A_1^* = A_2^*$ implies $A_1 = A_2$. Now the hypothesis $|G| \geq 2\alpha_k(S) + \kappa_k(S)$ translates to $|A_1| \leq |A_1^*|$ so that (16) implies $A_1^* \cap A_2^* \neq \emptyset$. The second part of Lemma 11 therefore applies and decomposing $A_1S \cap A_2S$ as

$$A_1S \cap A_2S = (A_1 \cap A_2) \cup (A_1 \cap \partial A_2) \cup (A_2 \cap \partial A_1) \cup (\partial A_1 \cap \partial A_2)$$

we have

$$|A_1S \cap A_2S| = \kappa_1 + 1 \leq |S|. \quad (17)$$

Let Z be a 2-atom of S^{-1} . By analogy to the previous case, we may assume that Z is left-periodic and the union of at least three cosets of its stabilizer by left translations (otherwise we are done.) Thus there are three distinct 2-atoms Z_1, Z_2, Z_3 of S^{-1} with an only common point z . By translating Z_1, Z_2, Z_3 if need be, we impose $z \in A_1^* \setminus A_2^*$.

By exactly the same argument with S^{-1} replacing S we deduce

$$|Z_1S^{-1} \cap Z_2S^{-1}| \leq |S|. \quad (18)$$

Let us now suppose, without loss of generality, that $\alpha_2(S) \leq \alpha_2(S^{-1})$. If this is not the case, simply switch S and S^{-1} .

Suppose first that A_1^* contains $Z_1 \cup Z_2$. This implies $Z_1^* \cap Z_2^* \supset A_1$, in particular $1 \in Z_1^* \cap Z_2^*$. Therefore we also have $Z_i^* \cap A_2 \neq \emptyset$ for $i = 1, 2$, so that Lemma 11 implies that $Z_i \cap A_2^*$ is a 2-fragment of S^{-1} or of cardinality 1. But we have chosen $z \in Z_i$ and $z \notin A_2^*$, so $Z_i \cap A_2^*$ is strictly included in Z_i , which means that $Z_i \cap A_2^*$ cannot be a 2-fragment of S^{-1} , because Z_i is a 2-atom. Therefore $|Z_i \cap A_2^*| \leq 1$ for $i = 1, 2$. But then, using $|A_1| \leq |Z_1|$ because we have supposed $\alpha_2(S) \leq \alpha_2(S^{-1})$,

$$|(Z_1 \cup Z_2) \cap A_1^* \setminus A_2^*| \geq 2|Z_1| - 3 \geq 2|A_1| - 3 > |A_1| - 1,$$

contradicting (16).

We may thus suppose that A_1^* contains at most one of the three atoms Z_1, Z_2, Z_3 . without loss of generality we may assume that each of Z_1 and Z_2 are not contained in A_1^* .

Now $Z_i \not\subset A_1^*$ implies $A_1 \not\subset Z_i^*$. A_1 is a 2-atom of S , Z_i is a 2-fragment of S , and since $\alpha_2(S) \leq \alpha_2(S^{-1})$, the intersection property of Theorem 4 implies $|A_1 \cap Z_i^*| \leq 1$.

This implies that A_1 intersects $Z_1S^{-1} \cap Z_2S^{-1}$ in at least $|A_1| - 2$ points. Moreover, since $z \in A_1^*$, we have $zS^{-1} \cap A_1 = \emptyset$. It follows that, by using (18),

$$|S| \geq |Z_1S^{-1} \cap Z_2S^{-1}| \geq |zS^{-1}| + |Z_1| - 2 \geq |S| + |A_1| - 2,$$

contradicting that $|A_1| = \alpha_2(S) > 2$. This completes the proof. \square

5 Periodic 2-atoms which are not subgroups

Throughout the section G is a finite group and S is a 2-separable generating set of G with $1 \in S$, $|S| \geq 3$, and $\kappa_2(S) = \kappa_1(S) = |S| - 1$. We assume that no 2-atom of S or of S^{-1} is a subgroup or has cardinality 2. By Lemmas 12 and 13 either S or S^{-1} has a 2-atom which is the union of two right cosets of some subgroup H ,

$$A = H \cup Ha.$$

Our main goal in this section is to prove the following result.

Theorem 14. *Let S be such that $|G| \geq 2\alpha_2(S) + \kappa_2(S)$ and $\kappa_2(S) = \kappa_1(S) = |S| - 1$. If S does not have 2-fragments that are subgroups and if $A = H \cup Ha$ is a 2-atom of S , then HS consists of the complement of exactly two right-cosets modulo H .*

Remark. The complement of HS must contain at least two cosets modulo H , otherwise we have $|G| = \alpha_2(S) + \kappa_2(S) + |H| < 2\alpha_2(S) + \kappa_2(S)$.

5.1 Reduction to the case when $|H| < 6$

Our approach will be the following: *left* multiplication by A defines a graph on the set of right-cosets Hx modulo the subgroup H . This graph (to be defined precisely below) is arc-transitive. Now the study of arc-transitive graphs shows them to have generally high connectivity. On the other hand, the condition $\kappa_2(S) = \kappa_1(S) = |S| - 1$ implies relatively low connectivity for the arc-transitive graph, and we will obtain a contradiction for all cases but the one mentioned in Theorem 14.

We first observe that $a^{-1}A \neq A$, since otherwise A is left-periodic by the subgroup generated by $H \cup \{a\}$ and therefore A coincides with this subgroup. Hence, since $1 \in A \cap a^{-1}A$, we have, by the intersection property for 2-atoms (Theorem 3):

$$H \cap a^{-1}Ha = \{1\}. \quad (19)$$

We also observe that

$$HS = AS, \quad (20)$$

since otherwise there is a full right-coset Hx contained in $AS \setminus HS$ and

$$|HS| - |H| \leq |AS| - |H| - |Hx| = |AS| - |A|,$$

which contradicts A being a 2-atom.

Let $X = \text{Cay}(G, A)$ be the *left* Cayley graph of A (arcs are $x \rightarrow \alpha x$, $\alpha \in A$). We define the (right) quotient graph X/H which has vertex set $\{Hx, x \in G\}$, the right cosets of H , and there is an arc $Hx \rightarrow Hy$ if and only if $HaHx \supset Hy$.

Lemma 15. *The graph X/H is vertex-transitive and arc-transitive. In particular X/H is regular and its degree is $|H|$.*

Proof. For each $z \in G$ we have $HaHx \supset Hy$ if and only if $HaHxz \supset Hyz$, so that the right translations $Hx \rightarrow Hxz$ are automorphisms of X/H and the set of right translations acts transitively on the vertex set of X/H .

Being vertex-transitive, to show that X/H is also arc-transitive, it suffices to show that the subset of $\text{Aut}(X/H)$ that leaves H invariant acts transitively on the set of neighbours of H . This follows by choosing left translations by $z \in H$.

In particular X/H is a regular graph. Observe that by (19), there are $|H|$ distinct right cosets in HaH . Therefore, X/H is regular of degree $|H|$. \square

The key observation in the use of the graph X/H is the following one. According to (20) and the fact that $\kappa_2(S) = |S| - 1$, we have

$$|HS| - |S| = |AS| - |S| = |A| - 1 = 2|H| - 1.$$

By looking at HS in the graph X/H we see that all arcs emanating from HS to $G \setminus HS$ lead to cosets Has for some $s \in HS \setminus S$. Hence, denoting by $e(HS)$ the number of arcs leading out from HS ,

$$e(HS) \leq 2|H| - 1. \quad (21)$$

At this point we will use some properties of arc-connectivity in arc-transitive graphs which can be found in [9]. The theory of atoms can be formulated for the arc-connectivity of graphs, in which setting it is somewhat simpler. For a subset C of the vertex set $V(Y)$ of a connected graph Y , we denote by $e(C)$ the set of arcs connecting a point in C to a point in $V(Y) \setminus C$. If $k \leq |V(Y)|/2$, we shall say that a subset of vertices C is k -separating if it has cardinality at least k and the set of vertices not in C has cardinality at least k . We shall say that the graph Y is k -separable if there exists a k -separating set. The k -arc connectivity $\lambda_k(Y)$ of Y is the minimum number of arcs leading out of a k -separating set, in other words:

$$\lambda_k(Y) = \min\{|e(C)| : k \leq |C| \leq |V(Y)| - k\}.$$

An arc k -fragment of Y is a set F of vertices with $e(F) = \lambda_k(Y)$, and an arc k -atom of Y is an arc k -fragment with minimum cardinality.

The next Lemma is Corollary 5 from [9].

Lemma 16. *Let Y be a connected arc-transitive graph with (out)degree d . If Y is k -separable then the arc k -atoms have cardinality at most $2k - 2$. If furthermore $k \leq d/3 + 1$, then every arc k -atom of Y has cardinality k . In particular,*

$$\lambda_k(Y) \geq dk - e_k(Y),$$

where $e_k(Y)$ is the largest number of arcs in a subgraph induced by a set of cardinality k . Moreover, the same conclusion holds for k up to $2d/3 + 1$ if G is antisymmetric (i.e. it has no 2-cycles).

We will apply Lemma 16 to obtain a contradiction with the hypothesis of Theorem 14 in the case when HS is a 3-separating set of X/H . This will yield the conclusion. We first show that we can limit ourselves to studying the case when X/H is a connected graph.

Note that X/H is not connected if and only if $\langle A \rangle$ is a proper subgroup of G . Consider the partition $S = S_1 \cup S_2 \cup \dots \cup S_m$ where S_i is the non-empty intersection of S with some right coset of G modulo $\langle A \rangle$. Since S generates G , we have $m > 1$ if X/H is not connected.

Claim 1. *We have $HS_i = \langle A \rangle S_i$ for all i , $1 \leq i \leq m$, except for one value of i .*

Proof. Every subgraph of X/H induced by HS_i is connected and arc-transitive, hence it has $\lambda_1 = |H|$ by Lemma 16 since the degree of X/H is $|H|$. Therefore if HS_i is 1-separating in its connected component, it has at least $|H|$ outgoing edges. But (21) implies that there can only be a single such HS_i . Note that we cannot have $HS_i = \langle A \rangle S_i$ for all i otherwise $AS = \langle A \rangle S$ which contradicts A being a 2-atom of S . \square

Without loss of generality, let S_1 be such that $HS_1 \neq \langle A \rangle S_1$ and $1 \in S_1$.

Claim 2. *We have $|AS_1| \leq |\langle A \rangle| - 2|H|$.*

Proof. If not, then $|AS_1| = |\langle A \rangle| - |H|$. But either $\langle A \rangle S \neq G$, and $|\langle A \rangle S| - |\langle A \rangle| \leq |AS| - |A|$ which contradicts the hypothesis of Theorem 14 that no fragment of S is a subgroup, or $\langle A \rangle S = G$, but then $|AS| = |G| - |H|$ so that $|G| = |AS| + |H| < |AS| + |A|$ meaning $|G| < 2\alpha_2(S) + \kappa_2(S)$ which also contradicts the hypothesis of Theorem 14. \square

Claim 3. *For every $i > 1$, we have $S_i = \langle A \rangle S_i$.*

Proof. If $S_i \neq \langle A \rangle S_i$ for some $i \neq 1$, then $|\langle A \rangle(S \setminus S_1)| > |S \setminus S_1|$ and Claim 1 implies

$$|S| - 1 = |AS| - |A| = |AS_1| - |A| + |\langle A \rangle(S \setminus S_1)| \quad (22)$$

so that we have

$$|AS_1| - |A| < |S_1| - 1.$$

In other words, $\kappa_1(S_1) < |S_1| - 1$. Furthermore, if B is a 2-fragment of S_1 , then we have

$$\begin{aligned} |BS| - |B| &= |BS_1| - |B| + |BS| - |BS_1| \\ &\leq |BS_1| - |B| + |\langle A \rangle(S \setminus S_1)| \end{aligned}$$

and by writing, since A is a 2-fragment of S ,

$$|AS| - |A| \leq |BS| - |B|,$$

we obtain that, applying (22) again,

$$|AS_1| - |A| \leq |BS_1| - |B|$$

so that A is also a 2-fragment of S_1 . Note that since B is a 2-fragment of S_1 , the two preceding inequalities must be equalities, so that B must also be a 2-fragment of S . By Claim 2 we have $|AS_1| \leq |\langle A \rangle| - |A|$, therefore $|\langle A \rangle| \geq 2\alpha_2(S_1) + \kappa_2(S_1)$, meaning that the intersection property must hold for 1-atoms of S_1 , and there is a 1-atom of S_1 that is a non-trivial subgroup. This subgroup must be a 2-fragment of S , contradicting the hypothesis of Theorem 14. \square

Finally we can now state:

Lemma 17. *Under the hypothesis of Theorem 14, replacing S by a right translate if need be, we have*

$$S = (S \cap \langle A \rangle) \cup (G \setminus \langle A \rangle).$$

Proof. The last claim means that $\langle A \rangle(S \setminus S_1) = S \setminus S_1$. By (20) we must have $|S_1| > 1$. Now if $S \setminus S_1 \neq G \setminus \langle A \rangle$, then $\langle A \rangle S \neq G$ and $|\langle A \rangle S| - |\langle A \rangle| < |S| - 1$, contradicting the hypothesis of Theorem 14. \square

The preceding lemma shows that $\kappa_1(S_1) = |S_1| - 1$ and that any 2-atom of S_1 is a 2-atom of S . We see therefore that it suffices to prove Theorem 14 with the additional hypothesis that A generates G . We will henceforth suppose this to be the case, so that we are dealing with a connected graph X/H .

We now work towards proving:

Proposition 18. *Suppose S has a 2-atom $A = H \cup Ha$ which is the union of two right cosets of some subgroup H and suppose that A defines a connected graph X/H . Then the subset of vertices of X/H defined by HS is not 3-separating.*

Remark. We have $|AS| - |A| = \kappa_2(S) > 0$, therefore $|AS| > 2|H|$ so that by (20) we have $|HS| > 2|H|$: hence HS in X/H consists of at least three vertices. Therefore HS defines a 3-separating set if and only if there are at least 3 vertices of X/H in the complement of HS .

Proof of Proposition 18 when $|H| \geq 6$. By Lemma 16 with $k = 3$, since the degree of X/H is $|H| \geq 6$ and $e_3(X/H) \leq 6$, we have

$$\lambda_3(X/H) \geq 3|H| - 6 > 2|H| - 1, \quad (23)$$

contradicting (21), so that HS cannot define a 3-separating subset of vertices of X/H . \square

We next study the small values of $|H|$.

5.2 Proof of Proposition 18 in the case $|H| = 3, 4, 5$

First note that an arc-transitive graph is either symmetric, or antisymmetric (meaning that if there is an arc from x to y then there is no arc from y to x). If X/H is antisymmetric, then $e_3(X/H) \leq 3$. If $|H| \geq 3$ then Lemma 16 applies for $k = 3$ and if HS is 3-separating in X/H we have $\lambda_3(X/H) \geq 3|H| - 3 \geq 2|H|$ which contradicts (21).

We are therefore left with the case when X/H is symmetric. We now rule this out.

Lemma 19. *If $A = H \cup Ha$ is a 2-atom of S , then the graph X/H cannot be symmetric.*

To prove Lemma 19 we will need the following easy fact.

Lemma 20. *When $a^2 \in H$, then for any $x \in G$, the set $T = Hx \cup a^{-1}Hx$ is stable by left multiplication by a , i.e. $aT = T$.*

Proof. We have $a^2 = h$ for some $h \in H$, hence $a = a^{-1}h$, therefore $aHx = a^{-1}Hx$. \square

Proof of Lemma 19. First note that, by changing a to ha for some $h \in H$ if need be, X/H is symmetric can be taken to mean that $a^2 \in H$.

Let C be the complement of HS in G . Since $HS = HaS$, no element of $a^{-1}C$ can be in S . Now, by Lemma 16 the arc 2-atoms of X/H have cardinality 2 and $\lambda_2(X/H) = 2|H| - 2$. By the remark following Theorem 14, the set HS must define a 2-separating subset of vertices of X/H , therefore $a^{-1}C$ intersects at least $2|H| - 2$ different H -cosets of HS . Since $|HS| = |AS| = |S| + 2|H| - 1$, we have that the complement of S equals

$$\overline{S} = C \cup a^{-1}C \cup E$$

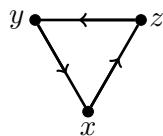
where E is either the empty set or a single element. But since, by Lemma 20, $a(C \cup a^{-1}C) = C \cup a^{-1}C$, we have that $\{1, a\}S = S$ (if $E = \emptyset$) or $|\{1, a\}S| = |S| + 1$ (if $|E| = 1$) so that A cannot be a 2-atom of S . \square

This concludes the proof of Proposition 18 in the case $|H| = 3, 4, 5$.

5.3 Proof of Proposition 18 in the case $|H| = 2$

We are now dealing with a graph X/H of degree $|H| = 2$ that must be antisymmetric by Lemma 19. We assume that HS defines a 3-separating set of X/H and work towards a contradiction. Inequality (21) tells us that we must have $\lambda_3 \leq 2|H| - 1 = 3$. Non-trivial arc-transitive graphs with these parameters do exist however and we do not have a contradiction directly. We shall therefore consider also arc 4-connectivity. By pure graph-theoretic arguments we will obtain that we must have $\lambda_4 \geq 4$ which will mean that HS cannot be 4-separating in X/H otherwise we would contradict (21). We will then be left with the case when HS is 3-separating but not 4-separating, meaning that HS consists of either three cosets modulo H or the complement of three cosets. We will then conclude the proof of Proposition 18 by excluding these cases separately.

Notice that if X/H contains no triangles (with any orientation) then subsets of 3 or 4 vertices must have at least 4 outgoing arcs, so that $\lambda_3 \geq 4$ since arc 3-atoms are of cardinality 3 or 4 (Lemma 16). We may therefore assume that every edge of X/H is contained (by arc-transitivity) in an oriented triangle:



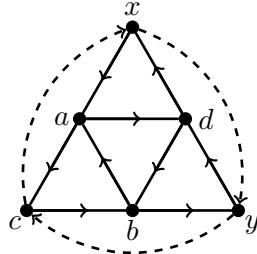
Indeed, both the in- and the out-neighbourhood of a vertex are vertex-transitive digraphs on two vertices, such a neighbourhood cannot therefore contain an edge otherwise it would contain also the reverse edge. Summarizing:

Lemma 21. *If $\lambda_3(X/H) \leq 3$ then every edge of X/H belongs to an oriented triangle. Furthermore arc 3-atoms are of cardinality 3 and are triangles.*

Let us denote call a K_4^* an antisymmetric graph on 4 vertices with 5 arcs (a K_4 with an edge removed).

Lemma 22. *If X/H is not an octahedron when orientation is removed, then it contains no K_4^* .*

Proof. Suppose the 4-vertex set $\{a, b, c, d\}$ forms a K_4^* , then orientations must be as below since triangles must be oriented.



Now we must have an arc (x, a) and an arc (b, y) with x, y outside the K_4^* . Note that we must have $x \neq y$ otherwise the outer neighbourhood of b would not be of degree 0. Note further that every arc, like arc (b, a) , must belong to two distinct triangles. For arc (x, a) , the two triangles can only be (x, a, c) and (x, a, d) , therefore we must have arcs (c, x) and (d, x) . Similarly, we must have arcs (y, c) and (y, d) , as in the picture. But then the picture can only be completed by arc (x, y) and we have an octahedron. \square

We now claim:

Proposition 23. *If X/H is 4-separable, then $\lambda_4 \geq 4$.*

Since 4-atoms have cardinality 4, 5 or 6 by Lemma 16, we prove Proposition 23 by showing that subsets of 4, 5 or 6 vertices have at least 4 outgoing arcs. We do this by looking at the non-oriented version of the graph, obtained from X/H by forgetting orientation, and showing that any subset of 4, 5, 6 vertices must have at least 8 outgoing edges. Note that we may suppose that the graph contains no K_4^* , since octahedrons are not 4-separable. Since we may also assume that every edge is included in a triangle (Lemma 21) we can claim:

Lemma 24. *If X/H is 4-separable and contains triangles, then the neighbourhood $N(x)$ of any vertex x is a 4-vertex graph of degree 1.*

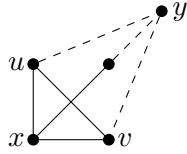
From the fact that X/H contains no K_4^* we obtain immediately:

Lemma 25. *If F is a 4-separating set of X/H with $|F| = 4$, then F has at least 8 outgoing edges.*

We now claim:

Lemma 26. *If F is a 4-separating set of X/H with $|F| = 5$, then F has at least 8 outgoing edges.*

Proof. Otherwise F contains at least 7 edges. If F contains a vertex x of degree 4 then the remaining 3 edges of F must be in $N(x)$. But $N(x)$ contains at most 2 edges by Lemma 24.

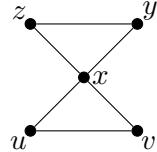


Therefore F cannot contain a vertex of degree 4 and since it contains 7 edges it must contain a vertex x of degree 3. Let y be the vertex of F that is not in $N(x)$.

There must be an edge between some two neighbours, say u and v , of x in F , otherwise we can put at most six edges in F . Since there cannot be any other edge in $N(x) \cap F$, the vertex y must be connected to all 3 neighbours of x in F . But then x, u, v, y make up a K_4^* , contradicting Lemma 22. \square

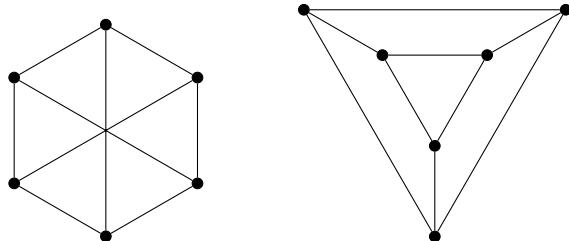
Lemma 27. *If F is a 4-separating set of X/H with $|F| = 6$, then F has at least 8 outgoing edges.*

Proof. Otherwise F contains at least 9 edges. Suppose first that F contains a vertex x of degree 4 in F . If there are no edges in $N(x)$ then F clearly cannot contain 9 edges, therefore the neighbourhood of x is of degree 1 and has the structure below:



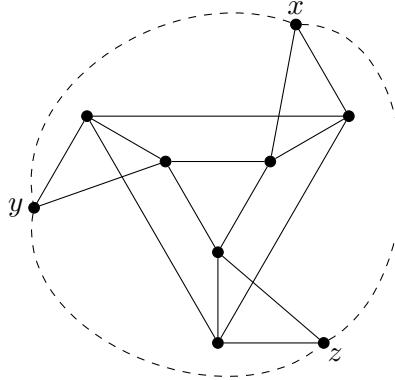
But then, to fit three extra edges in F from the remaining vertex w of F that is neither x nor in $N(x)$, we must connect w to either x and y or to u and v , which creates a K_4^* , contradicting Lemma 22.

Therefore, if F contains nine edges, the only possibility left is for F to be regular of degree 3. There are only two non-isomorphic graphs of degree 3 on six vertices which are:



The first graph contains no induced triangles: but we have seen that every edge of X/H must be included in some triangle. This implies in the case of the first graph that every one of its edges has both its endpoints connected to a common vertex. But since the degree of X/H is only 4, this vertex must be the same for every edge of F , which is not possible for a maximum degree 4. This excludes the first graph.

In the case of the second graph, again, because every edge must belong to a triangle, we must have the following picture:



Note that the vertices x, y and z must be distinct, because if two of them are equal we obtain a set of 8 vertices with two outgoing edges, which contradicts $\lambda_3 = 6$ ($\lambda_3(X/H) = 3$ translates to $\lambda_3 = 6$ when we remove arc-orientation).

Finally, from the structure of F where we see that every edge must not only belong to a triangle but also to a 4-cycle, we have that the only possibility is for the vertices x, y, z to be connected. But then every vertex is of degree 4 which means that we have described the whole graph X/H which has 9 vertices, so that F is not a 4-separable set. \square

Together Lemmas 25, 26 and 27 prove Proposition 23. To prove Proposition 18 we are just left with the cases when HS consists of either 3 cosets or the complement of 3 cosets. We shall need:

Lemma 28. *Without loss of generality, any triangle of X/H is made up of three cosets of the form Hx, Hax, Ha^2x .*

Proof. Set $H = \{1, h\}$. Let Hx be one of the cosets involved in the triangle. The coset Hx is connected by outgoing arcs to the cosets Hax and $Hahx$. If Hx is connected to $Hahx$ in the triangle, then rename x the group element hx to obtain that the triangle always contains an edge of the form $Hx \rightarrow Hax$. Now the coset Hax is connected by outgoing arcs to the cosets Ha^2x and $Hahax$. If Hax is connected to $Hahax$ in the triangle, then rename a the group element ha : this does not change the set $A = H \cup Ha$. \square

Proposition 29. *$A = H \cup Ha$ cannot be a 2-atom of S if HS consists of three right cosets modulo H .*

Proof. The cosets intersected by S must make up an arc 3-atom of X/H which is a triangle by Lemma 21 and of the form Hx, Hax, Ha^2x by Lemma 28. We have two cases, depending on the nature of the arc leading from Ha^2x to Hx .

- (a) We have $a^3 \in H$, in which case we must have $S = \{x, ax, a^2x\}$. But then $|\{1, a\}S| \leq 4 = |S| + |\{1, a\}| - 1$, so that $A = H \cup Ha$ cannot be a 2-atom of S .

(b) Setting $H = \{1, h\}$ we have $aha^2 \in H$. We cannot have $aha^2 = 1$, otherwise $a^3 \in H$ and we are back to the preceding case, therefore $aha^2 = h$. In this case we must have $S = \{x, ax, ha^2x\}$. Since $haha^2 = 1$ we have then $|\{1, ha\}S| \leq 4 = |S| + |\{1, ha\}| - 1$, so that $A = H \cup Ha$ cannot be a 2-atom of S . \square

Proposition 30. $A = H \cup Ha$ cannot be a 2-atom of S if the complement of HS consists of three right cosets modulo H .

Proof. By Lemma 21, the three cosets of the complement of HS must form a triangle so that, by Lemma 28, the complement of HS can be assumed to be equal to

$$Hx \cup Hax \cup Ha^2x.$$

From $HS = HaS$ we have that the complement of S must be equal to:

$$\overline{S} = a^{-1}(Hx \cup Hax \cup Ha^2x) \cup Hx \cup Hax \cup Ha^2x.$$

Now since there is an arc in X/H from Ha^2x to Hx , we have either $a^3 \in H$ or $aha^2 \in H$, where we write $H = \{1, h\}$.

(a) If $a^3 \in H$, then clearly $\overline{S}x^{-1}ax = \overline{S}$, so that $Sx^{-1}ax = S$, in other words S is stable by right multiplication by the subgroup generated by $x^{-1}ax$, but this contradicts $\kappa_1(S) = \kappa_1(S^{-1}) = |S| - 1$.

(b) If $aha^2 \in H$ then we must have $aha^2 = h$, because $aha^2 = 1$ implies $a^3 = h$ which brings us back to the preceding case. Now $aha^2 = h$ is equivalent to $ahaha = 1$, which implies that the cosets $H, Ha, Haha$ make up a triangle in X/H . But this means that the four cosets $H, Ha, Ha^2, Haha$ form a K_4^* in X/H . This in turn implies by Lemma 22 that X/H is an octahedron, which means that HS must consist of exactly three cosets modulo H . The result now follows from Proposition 29. \square

Together, Propositions 23, 29 and 30 complete the proof of Proposition 18. Given the remark after Theorem 14 stating that the complement of HS contains at least two cosets and the ensuing discussion leading to Proposition 18, this in turn proves Theorem 14.

6 Conclusion and Comments

6.1 Proof of Theorem 1

The three possibilities in the Theorem depend on the cardinality and structure of the 2-atom of S . We may assume that $\alpha_2(S) \leq \alpha_2(S^{-1})$. Also, if $\kappa_1(S) < |S| - 1$ then, by Theorem 5, the 1-atom of S is a subgroup H leading to case (ii). Thus we may assume that $\kappa_2(S) = \kappa_1(S) = |S| - 1$.

According to Proposition 10, either a 2-atom U of S has cardinality 2, which easily yields case (i), or U is periodic. Then, by Lemma 12 and Lemma 13, either there is a 2-fragment of S which is a subgroup, giving case (ii), or there is a 2-atom which is the union of two right cosets of some subgroup. In this last case, Theorem 14 gives case (iii): note that equality (19) that we have used throughout Section 5 gives the additional property $|HaH| = |H|^2$ mentioned in case (iii).

6.2 Examples when (i) and (ii) in Theorem 1 do not hold

As it has been already mentioned, the degenerate case in Theorem 1 (iii) can actually hold, without any of the two other cases being satisfied. We next give an infinite family of examples which illustrate this fact.

Let p be a prime and let q be an odd prime divisor of $p-1$. Let H_0 be the subgroup of order q of the multiplicative group of $\mathbb{Z}/p\mathbb{Z}$. Consider the semidirect product $G = \mathbb{Z}/p\mathbb{Z} \ltimes H_0$ defined by:

$$(x, h)(y, k) = (x + hy, hk).$$

Set $a = (1, 1)$ and $H = \{0\} \times H_0$. We have $a^{-1} = (-1, 1)$ and:

$$\begin{aligned} Ha &= \{(h, h), h \in H_0\} \\ a^{-1}H &= \{(-1, h), h \in H_0\} \\ a^{-1}Ha &= \{(-1 + h, h), h \in H_0\} \end{aligned}$$

Let $A = H \cup Ha$ and

$$B = H \cup Ha \cup a^{-1}H \cup a^{-1}Ha$$

and set $S = G \setminus B$.

Since $-1 \notin H_0$ (H_0 has odd order) we have $a^{-1}H \cap Ha = \emptyset$ and $a^{-1}Ha \cap H = \{(0, 1)\}$. Therefore

$$(H \cup Ha) \cap (a^{-1}H \cup a^{-1}Ha) = \{(0, 1)\}$$

and $|S| = |G| - 4|H| + 1$. We have $AS = G \setminus A$, hence

$$|AS| = |S| + |A| - 1.$$

Note that $B = B^{-1}$, so that $S = S^{-1}$. The subgroup H does not satisfy condition (ii) of Theorem 1 since we have

$$|HS| = |S| + 2|H| - 1.$$

Let us check the other proper subgroups of G . Since $|G| = pq$ every proper subgroup is cyclic and

$$(x, h)^i = (x(1 + h + \cdots + h^{i-1}), h^i)$$

so that (x, h) is of order p when $h = 1$ and of order q otherwise. There is therefore just one subgroup of order p namely

$$G_p = \{(x, 1), x \in \mathbb{Z}/p\mathbb{Z}\}.$$

It is straightforward to check that for every $b \in B$,

$$G_p b \cap S \neq \emptyset$$

hence $G_p S = G$. It is also straightforward to check that all subgroups of order q coincide with the set of conjugate subgroups

$$K_x = (x, 1)^{-1} H(x, 1) = \{(x(h-1), h), h \in H_0\}$$

where x ranges over $\mathbb{Z}/p\mathbb{Z}$. It is again readily checked that for $x \neq 0, 1$ and for every $b \in B$,

$$K_x b \cap S \neq \emptyset$$

so that $K_x S = G$. For $x = 0$ we have $K_0 = H$ and for $x = 1$, $K_1 = a^{-1}Ha$ in which case

$$|K_1 S| = |S| + 2|K_1| - 1.$$

Since $S^{-1} = S$ we have shown that case (ii) of Theorem 1 does not hold.

For case (i) of Theorem 1 to hold, there must exist a group element $r \neq 1$ such that $|\{1, r\}S| = |S| + 1$. But this means that we would have $|\{1, r\}B| = |B| + 1$ as well, and this is readily excluded for all r .

Hence only case (iii) of Theorem 1 can hold for this group G and this subset S .

Note that if we have $q = (p - 1)/2$ (so that q is a Sophie Germain prime), then we have $|S| = |G| - 4|H| + 1 = |G| - 2\sqrt{2}|G|^{1/2}(1 - O(q^{-1/2}))$. In particular, the condition $|S| \leq |G| - 4|G|^{1/2}$ in Corollary 2 cannot be improved upon significantly.

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